Solutions to boundary error problem of HASM

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Abstract

High accuracy surface modeling method (HASM) is more accurate than the classic methods and has been used widely in many areas . However, HASM suffers from the boundary error problem which significantly lower the accuracy of HASM in the whole research area and makes HASM often slightly better than other methods. In this research, we give a new method to improve HASM's simulation skills especially at the border of the research areas.Numerical tests demonstrate efficiency of the proposed method.

Keyword HASM; error; accuracy loss; boundary

1 Introduction

Error problem of surface modeling method has become a major concern since the 1960s (Stott 1977; Li and Zhu 2000). Although several improvement techniques were proposed, the error problems still exist that have long troubled surface modeling. To find a solution for the error problems that produced by geographical information systems (Wise 2000; Yue et al. 2002), a method of high accuracy surface modeling (HASM) in terms of the fundamental theorem of surface theory was proposed by Yue (2011). Numerical and real-world tests show that HASM is much more accurate than the classic methods, such as Kriging, IDW and splines (Yue and Du 2006; Yue and Wang 2010; Yue et al. 2007; Yue and Song 2008; Yue et al. 2010).

Despite its higher accuracy than other classic methods, however, HASM suffers from the boundary error problem which significantly lower the accuracy of HASM in the whole research area and makes HASM often slightly better than other methods. Studies show that the accuracy on the boundary has a great influence on the accuracy of HASM (Yue, 2011). Several efforts have attempted to solve the boundary error problem (Chen et al. 2010, Yue, 2011). But these methods are either consulting to other interpolation method or resorting to other partial difference equations, such as Laplace equation, to give the boundary value, which make HASM become more complex. Besides, the introduction of Laplace equation is unreasonable and the physical mean of this is not clear.

In this paper, we give the solution of the boundary error problem based on the fundamental basis of HASM, which make HASM be simple and more accuracy. We research the high order difference schemes of HASM on the boundary points and change the boundary problem to the initial problem, and then subsequently improved by the iteration process. We demonstrate this method by using Gauss synthetic surface and compare it with the current method.

2 Theoretical basis of HASM

As an innovative method, high accuracy surface modeling (HASM) is based on the fundamental theorem of surfaces which makes sure that a surface is uniquely defined by the first and the second fundamental coefficients (Henderson, 1998). HASM method, combined with Gauss-Codazzi equations, divides the simulated areas in an uniform orthogonal way and establishes the corresponding difference equations with finite difference method. We then solve this problem under the restriction of the sample data (Yue et al., 2004).

The optimum formulation of HASM. The optimal equations of HASM can be formulated as Yue and Du (2006),

$$\begin{cases} f_{xx} = \Gamma_{11}^{1} f_{x} + \Gamma_{11}^{2} f_{y} + L(EG - F^{2})^{-1/2} \\ f_{yy} = \Gamma_{22}^{1} f_{x} + \Gamma_{22}^{2} f_{y} + N(EG - F^{2})^{-1/2}, \end{cases}$$
(1)

where, $E = 1 + f_x^2$, $F = f_x f_y$, $G = 1 + f_y^2$, $L = f_{xx} / \sqrt{1 + f_x^2 + f_y^2}$, $N = f_{yy} / \sqrt{1 + f_x^2 + f_y^2}$,

$$\Gamma_{11}^{1} = 1/2 \big(GE_{x} - 2FF_{x} + FE_{y} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - FE_{x} \big) (EG - F^{2})^{-1}, \\ \Gamma_{11}^{2} = 1/2 \big(2EF_{x} - EE_{y} - EE_{y} \big) (EG - FE_{x} - FE_{y} \big) (EG - FE_{$$

$$\Gamma_{22}^{1} = 1/2 (2GF_{y} - GG_{x} - FG_{y})(EG - F^{2})^{-1}, \Gamma_{22}^{2} = 1/2 (EG_{y} - 2FF_{y} + FG_{x})(EG - F^{2})^{-1}.$$

Let $\{(x_i, y_j)| 0 \le i \le I + 1, 0 \le j \le J + 1\}$ be the computational grids and h be the grid size in x and y directions. The discrete forms of f_x , f_y and f_{xx} , f_{yy} at the inner points are as follows:

$$(f_{x})_{(i,j)} \approx \begin{cases} \frac{f_{1,j}-f_{0,j}}{h} & i = 0\\ \frac{f_{i+1,j}-f_{i-1,j}}{2h} & i = 1, \cdots, I \\ \frac{f_{l+1,j}-f_{l,j}}{h} & i = I+1 \end{cases} \quad (f_{xx})_{(i,j)} \approx \begin{cases} \frac{f_{0,j}+f_{2,j}-2f_{1,j}}{h^{2}} & i = 0\\ \frac{f_{i-1,j}-2f_{i,j}+f_{i+1,j}}{2h^{2}} & i = 1, \cdots, I \\ \frac{f_{l+1,j}-f_{l,j}}{h^{2}} & i = I+1 \end{cases}$$

$$(f_{y})_{(i,j)} \approx \begin{cases} \frac{f_{i,1}-f_{i,0}}{h} & j = 0\\ \frac{f_{i,j+1}-f_{i,j-1}}{2h} & j = 1, \cdots, J \\ \frac{f_{i,j+1}-f_{i,j}}{h} & j = J+1 \end{cases} \quad (f_{yy})_{(i,j)} \approx \begin{cases} \frac{\frac{f_{i,0}+f_{i,2}-2f_{i,1}}{h^{2}} & j = 0\\ \frac{f_{i,j-1}-2f_{i,j}+f_{i,j+1}}{2h^{2}} & j = 1, \cdots, J \\ \frac{f_{i,j+1}-f_{i,j}}{h^{2}} & j = J+1 \end{cases}$$

By taking into account the sampling points, we obtain the equations of HASM for the inner points:

$$\begin{cases} \frac{f_{i+1,j}^{n+1} - 2f_{i,j}^{n+1} + f_{i-1,j}^{n+1}}{h^2} = (\Gamma_{11}^1)_{i,j}^n \frac{f_{i+1,j}^n - f_{i-1,j}^n}{2h} + (\Gamma_{11}^2)_{i,j}^n \frac{f_{i,j+1}^n - f_{i,j-1}^n}{2h} + \frac{L_{ij}^n}{\sqrt{E_{i,j}^n + G_{i,j}^n - 1}} \\ \frac{f_{i,j+1}^{n+1} - 2f_{i,j}^{n+1} + f_{i,j-1}^{n+1}}{h^2} = (\Gamma_{22}^1)_{i,j}^n \frac{f_{i+1,j}^n - f_{i-1,j}^n}{2h} + (\Gamma_{22}^2)_{i,j}^n \frac{f_{i,j+1}^n - f_{i,j-1}^n}{2h} + \frac{N_{ij}^n}{\sqrt{E_{i,j}^n + G_{i,j}^n - 1}} \\ f_{i,j} = \tilde{f}_{i,j} \quad , \quad (x_i, y_j) \in \Phi \end{cases}$$

$$(3)$$

where,

$$\begin{split} E_{i,j}^{n} &= 1 + (\frac{f_{i+1,j} - f_{i-1,j}}{2h})^{2}, \ \ G_{i,j}^{n} = 1 + (\frac{f_{i,j+1} - f_{i,j-1}}{2h})^{2}, \ \ L_{i,j}^{n} = \frac{\frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{2h^{2}}}{\sqrt{1 + (\frac{f_{i+1,j} - f_{i-1,j}}{2h})^{2} + (\frac{f_{i,j+1} - f_{i,j-1}}{2h})^{2}}}, \\ N_{i,j}^{n} &= \frac{\frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{2h}}{\sqrt{1 + (\frac{f_{i+1,j} - f_{i-1,j}}{2h})^{2} + (\frac{f_{i,j+1} - f_{i,j-1}}{2h})^{2}}}}{\sqrt{1 + (\frac{f_{i+1,j} - f_{i-1,j}}{2h})^{2} + (\frac{f_{i,j+1} - f_{i,j-1}}{2h})^{2}}}, \ \ (\Gamma_{11}^{1})_{i,j}^{n} &= \frac{E_{i+1,j} - E_{i-1,j}}{4hE_{i,j}}, \ \ (\Gamma_{12}^{2})_{i,j}^{n} &= \frac{E_{i,j+1} - E_{i,j-1}}{4hG_{i,j}}, \\ (\Gamma_{22}^{1})_{i,j}^{n} &= -\frac{G_{i+1,j} - G_{i-1,j}}{4hE_{i,j}}, \ (\Gamma_{22}^{2})_{i,j}^{n} &= \frac{G_{i,j+1} - G_{i,j-1}}{4hG_{i,j}}, \ \ \Phi &= \{(x_{i}, y_{j}, \tilde{f}_{i,j}) | 1 \leq i \leq I, \ 1 \leq j \leq J\} \end{split}$$

is a set of sample points, n represents the number of iterations.

Currently, we just process the inner points of the research area and let the values of boundary grid points equal to the initial values. That is, $f_{0,j} = f_{0,j}^{(0)}, f_{I+1,j} = f_{I+1,j}^{(0)}, j = 0, ..., J+1, f_{0,j}^{(0)}$ and $f_{I+1,j}^{(0)}$ are the initial value at the boundary points which can be obtained by other simple method, such as linear interpolation method. The value of $f_{0,j}$, $f_{I+1,j}$, j = 0, ..., J + 1 will not be updated in the run process of HASM just as shown in equations (3). This undoubtedly reduces the accuracy of HASM especially at the border of the research area. Then, we first give the new discrete form of f_x , f_y and f_{xx} , f_{yy} as follows,

$$(f_x)_{(i,j)} \approx \begin{cases} \frac{-3f_{0,j} + 4f_{1,j} - f_{2,j}}{2h} & i = 0\\ \frac{f_{i+1,j} - f_{i-1,j}}{2h} & i = 1, \dots I, \\ \frac{3f_{I+1,j} - 4f_{I,j} + f_{I-1,j}}{2h} & i = I+1 \end{cases}$$

$$(f_{xx})_{(i,j)} \approx \begin{cases} \frac{2f_{0,j} - 5f_{1,j} + 4f_{2,j} - f_{3,j}}{h^2} & i = 0\\ \frac{-f_{i+2,j} + 16f_{i+1,j} - 30f_{i,j} + 16f_{i-1,j} - f_{i-2,j}}{12h^2} & i = 1, \dots I \\ \frac{2f_{I+1,j} - 5f_{I,j} + 4f_{I-1,j} - f_{I-2,j}}{h^2} & i = I + 1 \end{cases}$$

$$(f_y)_{(i,j)} \approx \begin{cases} \frac{-3f_{i,0} + 4f_{i,1} - f_{i,2}}{2h} & j = 0\\ \frac{f_{i,j+1} - f_{i,j-1}}{2h} & j = 1, \dots J \\ \frac{3f_{i,J+1} - 4f_{i,J} + f_{i,J-1}}{2h} & j = J+1 \end{cases}$$

$$(f_{yy})_{(i,j)} \approx \begin{cases} \frac{2f_{i,0} - 5f_{i,1} + 4f_{i,2} - f_{i,3}}{h^2} & j = 0\\ \frac{-f_{i,j+2} + 16f_{i,j+1} - 30f_{i,j} + 16f_{i,j-1} - f_{i,j-2}}{12h^2} & j = 1, \dots \\ \frac{2f_{i,J+1} - 5f_{i,J} + 4f_{i,J-1} - f_{i,J-2}}{h^2} & j = J+1 \end{cases}$$

For the boundary points, $f_{0,j}$, $f_{I+1,j}$, j = 0, ..., J+1, we fist let them equal to the initial values and then updated them via the process the HASM, which are different from the existed version. Because we not only employ the higher discrete forms of the derivative term but also updated the values of the boundary points through the process of HASM. And, for the inner points , equations (3) can be translated into

$$\begin{pmatrix} -f_{i+2,j}^{(n+1)} + 16f_{i+1,j}^{(n+1)} - 30f_{i,j}^{(n+1)} + 16f_{i-1,j}^{(n+1)} - f_{i-2,j}^{(n+1)} \\ 12h^2 \\ -f_{i,j+2}^{(n+1)} + 16f_{i,j+1}^{(n+1)} - 30f_{i,j}^{(n+1)} + 16f_{i,j-1}^{(n+1)} - f_{i,j-2}^{(n+1)} \\ -f_{i,j-2}^{(n+1)} + 16f_{i,j+1}^{(n+1)} - 30f_{i,j}^{(n+1)} + 16f_{i,j-1}^{(n+1)} - f_{i,j-2}^{(n+1)} \\ -f_{i,j-2}^{(n+1)} - f_{i,j-2}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} - f_{i,j-2}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} - f_{i,j-2}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} - f_{i,j-1}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} - f_{i,j-1}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} - f_{i,j-1}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} + f_{i,j-1}^{(n)} - f_{i,j-1}^{(n)} - f_{i,j-1}^{(n)} \\ -f_{i,j}^{(n)} - f_{i,j-1}^{(n)} - f_{i,j-1}^{(n)}$$

3 Results

Gauss synthetic surface is used to validate the new version of HASM. The formula of this surface is expressed as:

$$f(x,y) = 3(1-x)^2 e^{-x^2 - (y+1)^2} - 10\left(\frac{x}{5} - x^3 - y^5\right) e^{-x^2 - y^2} - \frac{e^{-(x+1)^2 - y^2}}{3},$$

the computational domain is $[-3,3]\times[-3,3]$ and -6.5510 < f(x,y) < 8.1062. The grid size is denoted by h and the sampling interval is 2h in this research.

We compare the results of the proposed method, denoted as HASM2, and the current version of HASM denoted by HASM1. Root mean square error (RMSE) is used to evaluate their performances. The formula of RMSE is

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N} (f_{ij} - \hat{f}_{i,j})^2}{N}},$$

where $f_{i,j}$ is the true value at point (x_i, y_j) ; $\hat{f}_{i,j}$ is the simulated value; N is the number of validation points.

The values of RMSE at the border are shown in Table1. Denote the outmost edge of the computational grids as 0, and from the outside in, the label is added by one. For different boundary layers, we compare four directions: above, below, left, and right.

Table 1 Estimation errors of HASM1, HASM2 at the boundary of calculating area

(grid number 2209)

				-			
	0		1		2		
	upper	0.0617	upper	0.7198	upper	1.221	
HASM1	lower	0.0501	lower	0.7921	lower	1.3290	
	left	0.0093	left	0.3155	left	0.4096	

	right	0.0071	right	0.3534	right	0.4898	
HASM2	0		1		2		
	upper	0.0263	upper	0.3665	upper	0.9001	
	lower	0.0399	lower	0.4816	lower	0.9186	
	left	0.0265	left	0.0496	left	0.2018	
	right	0.0266	right	0.0501	right	0.1997	

Results (Table 1) reveal that the estimation errors on different layers produced by HASM1are considered a significant difference and the oscillation are occurred. On the layers '1' and '2', the errors of HASM2 are smaller than HASM1 which make the surface 'smoother'.

4 Conclusions

We solve the boundary error problem of HASM and give a higher accuracy version of it. The method proposed in this research does not depend on other nonlinear partial equations and just changes the code of HASM. It is simple and efficiency. Numerical tests show that modified HASM removes the oscillation at the border and the interpolation accuracy of it is higher than the current version.

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